Lecture 11 : Linear programming and Integer Linear programming

This lecture presents linear programming, one of the most classical optimization techniques. After introducing its formulation and one important variant (Integer Linear Programming), we will explore it through the lens of constrained optimization theory, notably its dual function, the KKT condition and the strong duality theorem (for linear programming). In the last section, we study how to use apply linear programming to design approximation algorithms for NP-hard problems.

1 Linear programming - problem formulation and duality

1.1 Problem formulation, examples and variants

Linear programming admits the following standard formulation:

$$\begin{aligned} \underset{x\in\mathbb{R}^n}{\text{Minimize}} \quad & c^\top x \\ \text{subject to:} \quad & \mathbf{A}x \geq b, \\ & x \geq 0. \end{aligned} \tag{LP}$$

where $\mathbf{A} \in \mathbb{R}^{m\times n}, b \in \mathbb{R}^m, c \in \mathbb{R}^n$ are given.

Linear programming is a very natural formulation for economic and operational modeling (e.g., the diet problem, logistics). Consider several examples of linear programming:

Example 1.1 (Feasible instance)**.** Consider the following problem:

$$\begin{aligned} \text{Minimize} \quad & x_1 + 2x_2 \\ \text{subject to:} \quad & 2x_1 + x_2 \geq 3, \\ & 2x_1 - x_2 \geq 2, \\ & x_1, x_2 \geq 0. \end{aligned}$$

The problem is feasible (show a diagram) and have an optimal solution $(x_1, x_2) = (1.5, 0)$. If the infimum of a linear programming instance is finite, then it is attained (using the Fourier-Motzkin argument).

Example 1.2 (Unbounded problem)**.** Consider the following problem:

$$\begin{aligned} \text{Minimize} \quad & -3x_1 - 2x_2 \\ \text{subject to:} \quad & 2x_1 + x_2 \geq 3, \\ & x_1 + x_2 \geq 2, \\ & x_1, x_2 \geq 0. \end{aligned}$$

By choosing $x_1 = x_2 = t \geq 1$, the objective value is given by $-5t$ and we can make t go to infinity.

Example 1.3 (Infeasible problem)**.** Consider the following problem:

$$\begin{aligned} \text{Minimize} \quad & 0 \\ \text{subject to:} \quad & 2x_1 + x_2 \geq 3, \\ & -2x_1 - x_2 \geq -2, \\ & x_1, x_2 \geq 0. \end{aligned}$$

Therefore, given a linear programming instance, one has exactly three possibilities: either the problem admits at least one minimizer, or it is unbounded, or it is infeasible.

Note that in reality, one can meet several variant of linear programming such as:

$$\begin{aligned} \underset{x \in \mathbb{R}^n}{\text{Minimize}} \quad & c^\top x \\ \text{subject to:} \quad & \mathbf{A}x \geq b. \end{aligned} \tag{1}$$

where we do not have positivity constraints or:

$$\begin{aligned} \underset{x \in \mathbb{R}^n}{\text{Minimize}} \quad & c^\top x \\ \text{subject to:} \quad & \mathbf{A}_1 x \geq b_1, \\ & \mathbf{A}_2 x = b_2, \\ & x \geq 0. \end{aligned} \tag{2}$$

where we also have equality constraints. Note that one can always reduce the variants to (LP). For example, one can introduce x_+ and x_- and write (1) equivalent to:

$$\begin{aligned} \underset{x \in \mathbb{R}^n}{\text{Minimize}} \quad & c^\top x_+ + c^\top x_- \\ \text{subject to:} \quad & \mathbf{A}(x_+ - x_-) \geq b, \\ & x_+, x_- \geq 0. \end{aligned} \tag{3}$$

Similarly, we can replace equality constraints $\mathbf{A}x = b$ by two inequalities: $\mathbf{A}x \geq b$ and $-\mathbf{A}x \geq -b$.

1.2 Dual problem

Given a linear programming problem, we compute its dual function using the Larangian function:

$$\mathcal{L}(x, \lambda, \nu) = c^\top x - \lambda^\top(\mathbf{A}x - b) - \nu^\top x = (c - \mathbf{A}^\top \lambda - \nu)^\top x + b^\top \lambda.$$

The dual function is given by:

$$\inf_{x \in \mathbb{R}} \mathcal{L}(x, \lambda, \nu) = \begin{cases} b^\top \lambda, & c = \mathbf{A}^\top \lambda + \nu \\ -\infty, & \text{otherwise} \end{cases}$$

Thus, the dual optimization problem of (LP) is:

$$\begin{aligned} \underset{y \in \mathbb{R}^n}{\text{Maximize}} \quad & b^\top y \\ \text{subject to:} \quad & \mathbf{A}^\top y \leq c, \\ & y \geq 0. \end{aligned} \tag{D-LP}$$

Therefore, the dual of a linear programming problem is also a linear programming one. We can verify that the dual of (D-LP) is (LP).

Example 1.4. The dual optimization of Example 1.1 is:

$$\begin{aligned} \text{Maximize} \quad & 3y_1 + 2y_2 \\ \text{subject to:} \quad & 2y_1 + 2y_2 \leq 1, \\ & y_1 - y_2 \leq 2, \\ & y_1, y_2 \geq 0. \end{aligned} \tag{4}$$

Note that one can also derive by interpreting (D-LP) as an attempt to maximize the lower-bound of (LP). Since the constraints of (LP) are linear, the linear constraint qualifications (LCQ) are satisfied and KKT is the necessary condition for a optimal solution. Let us remind the version of KKT for (LP).

Proposition 1.5 (KKT conditions)**.** *If $x^\star$ is an optimal solution of* (LP)*, there exists $\lambda, \nu \geq 0$ such that:*

$$c - \mathbf{A}^\top \lambda - \nu = 0 \tag{Stationarity}$$
$$\mathbf{A}x \geq b, x \geq 0 \tag{Primal feasibility}$$
$$\lambda \geq 0, \nu \geq 0 \tag{Dual feasibility}$$
$$\lambda^\top (\mathbf{A}x - b) = 0, \nu^\top x = 0. \tag{Complementary slackness}$$

The beautiful thing about (LP) and (D-LP) is: if (LP) is feasible, then strong duality holds, i.e, the optimal values of (LP) and (D-LP) are equal.

Theorem 1.6 (Strong duality of (LP))**.** *If* (LP) *is feasible, then the optimal values of* (LP) *and* (D-LP) *are equal.*

Proof. Let $p^\star$ and $d^\star$ be the optimal values of (LP) and (D-LP) respectively. By weak duality theorem, we always have: $p^\star \geq d^\star$. It is, thus, sufficient to prove that $d^\star \geq p^\star$. Since (LP) is feasible, it must admit an optimal solution $x^\star$. By the KKT conditions, there exists $\lambda, \nu \geq 0$ such that the conditions in Proposition 1.5 hold. We consider $\lambda \geq 0$, which is feasible for (D-LP) because:

$$\mathbf{A}^\top \lambda = c - \nu \leq c.$$

Moreover,

$$d^\star \geq b^\top \lambda = (x^\star)^\top \mathbf{A}^\top \lambda = (x^\star)^\top (c + \nu) = (x^\star)^\top c + (x^\star)^\top \nu = (x^\star)^\top c = p^\star.$$

This conclude the proof. □

We have the following important theorem for linear programming:

Theorem 1.7 (Primal-dual relation)**.** *Given a primal (P) and dual (D) linear programming problems, we have:*

1. *If (P) is feasible, then (D) is also feasible and their optimal values are equal.*
2. *If (P) is unbounded, then (D) is infeasible.*
3. *If (D) is unbounded, then (P) is infeasible.*

2 Integer Linear Programming

Integer Linear Programming (ILP) is nearly identical to Linear Programming. The only difference is that ILP also constraints several variables to be integers. In general, we can consider the following general formulation:

$$\begin{aligned} \underset{x\in\mathbb{R}^n}{\text{Minimize}} \quad & c^\top x \\ \text{subject to:} \quad & \mathbf{A}x \geq b, \\ & x \geq 0, \\ & x_i \in \{0,1\}, \forall i \in \mathcal{I}. \end{aligned} \tag{ILP}$$

Since integers can be represented by binary digits, we can use binary constraints, WLOG.

Example 2.1 (Minimum Vertex Covering)**.** Consider a undirected graph $G = (V, E)$. We want to find a vertex covering set S, i.e. $\forall (i,j) \in E$, either $i \in S$ or $j \in S$. The corresponding ILP formulation is:

$$\begin{aligned} \underset{x\in\mathbb{R}^{|V|}}{\text{Minimize}} \quad & \sum_{i\in V} x_i \\ \text{subject to:} \quad & x_i + x_j \geq 1, \forall (i,j) \in E, \\ & x_i \in \{0,1\}, \forall i \in V. \end{aligned} \tag{Vertex-Cover}$$

Example 2.2 (Marriage problem)**.** Consider n men $\{M_i, i = 1,\dots,n\}$ and m women $\{W_i, i = 1,\dots,m\}$. Let E be the set of pairs (i,j) such that if $(i,j) \in S$ if the ith man and jth woman can marry each other. The corresponding ILP formulation is:

$$\begin{aligned} \underset{x\in\mathbb{R}^{m\times n}}{\text{Maximimze}} \quad & \sum_{1\leq i\leq n, 1\leq j\leq m} x_{i,j} \\ \text{subject to:} \quad & x_{i,j} = 0, \forall (i,j) \notin E, \\ & x_{i,j} \in \{0,1\}, \forall 1 \leq i \leq m, 1 \leq j \leq n. \end{aligned} \tag{Marriage-Problem}$$

2.1 Applications of linear programming

Linear programming can provide a lower-bound for (ILP). The most natural one is:

$$\begin{aligned} \underset{x\in\mathbb{R}^n}{\text{Minimize}} \quad & c^\top x \\ \text{subject to:} \quad & \mathbf{A}x \geq b, \\ & x \geq 0, \\ & 0 \leq x_i \leq 1, \forall i \in \mathcal{I}. \end{aligned} \tag{Relaxed-ILP}$$

In general, (Relaxed-ILP) only provides a strict lower-bound for (ILP). With very few exceptions (such as (Marriage-Problem)) that (Relaxed-ILP) gives a lowerbound matching the optimal value of (ILP) (note that the set of minimizers of the relaxed version can be strictly bigger though).

The relaxation (Relaxed-ILP) can be exploited in many different ways: the most standard one is in branch-and-bound methods where the relaxation can be solved much more efficiently and helps us to fathom a node in the branch-and-bound tree faster. In this lecture, we will consider another type of applications of (Relaxed-ILP): to design approximation algorithms.

Consider the weighted variant of (Vertex-Cover), i.e.,

$$\begin{aligned} \underset{x\in\mathbb{R}^{|V|}}{\text{Minimize}} \quad & \sum_{i\in V} w_i x_i \\ \text{subject to:} \quad & x_i + x_j \geq 1, \forall (i,j) \in E, \\ & x_i \in \{0,1\}, \forall i \in V. \end{aligned} \tag{WVC}$$

where $w_i \geq 0, i \in V$ are positive weights.

Let $v^\star$ be the optimal value of (WVC). In the following, we provide a method to find a feasible x such that $w^\top x \leq 2v^\star$, i.e., a 2-approximation solution of (WVC).

Consider the relaxation of (WVC) given by:

$$\begin{aligned} \underset{x \in \mathbb{R}^{|V|}}{\text{Minimize}} \quad & \sum_{i \in V} w_i x_i \\ \text{subject to:} \quad & x_i + x_j \geq 1, \forall (i,j) \in E, \\ & 0 \leq x_i \leq 1, \forall i \in V. \end{aligned} \tag{Relaxed-WVC}$$

Let $x^\star$ be an optimal solution of Equation (Relaxed-WVC) (which can be calculated efficiently). Constructing a solution of the form:

$$\bar{x}_i = \begin{cases} 1, & \text{if } x_i \geq 1/2, \\ 0, & \text{otherwise} \end{cases}$$

In fact, $\bar{x}$ is feasible for (WVC). Indeed, since $x_i^\star + x_j^\star \geq 1, \forall (i,j) \in E$, at least one of $x_i^\star$ or $x_j^\star$ is at least $1/2$ and hence the argument. Finally, we have:

$$\bar{x}^\top w \leq 2(x^\star)^\top w \leq 2v^\star.$$