



Tutorial 9 : Nonsmooth Optimization - Proximal algorithms

Exercise 1 (Coercivity of strongly convex function). Prove that if a function f is μ -strongly convex ($\mu > 0$), then f is coercive, i.e., $\lim_{\|x\| \rightarrow +\infty} f(x) = +\infty$.

Exercise 2 (Properties of proximal operators). Assume in the following that the proximal operators are well-defined. Prove that:

1. Separable sum: if $f(x, y) = \varphi(x) + \phi(y)$, then: $\text{prox}_f(x, y) = (\text{prox}_\varphi(x), \text{prox}_\phi(y))$.
2. Affine transformations:
 - (a) If $f(x) = \alpha g(x) + \beta$ with $\alpha > 0$, then $\text{prox}_f(x) = \text{prox}_{\alpha g}(x)$.
 - (b) If $f(x) = g(\alpha x + \beta)$ with $\alpha \neq 0$, then $\text{prox}_f(x) = \frac{1}{\alpha}(\text{prox}_{\alpha^2 f}(\alpha x + \beta) - \beta)$.
 - (c) If $f(x) = g(x) + a^\top x + b$, then $\text{prox}_f(x) = \text{prox}_g(x - a)$.

Exercise 3 (Proximal operators). Compute the proximal operators of the following functions:

1. Norm ℓ_1 : $f(x) = \lambda \|x\|_1, \lambda > 0$
2. Norm ℓ_2 : $f(x) = \lambda \|x\|_2, \lambda > 0$
3. Group Lasso: $f(x) = \sum_{i=1}^K \lambda_i \|x_{\mathcal{I}_i}\|_2$ where $\mathcal{I}_i, i = 1, \dots, k$ form a partition of $\{1, \dots, d\}$ and $x_{\mathcal{I}_i}$ is the vector x restricted to the set of index in \mathcal{I}_i .
4. Pseudo-norm ℓ_0 : $f(x) = \lambda \sum_{i=1}^d \mathbf{1}_{x_i \neq 0}$ (attention: nonconvex function).
5. Hinge loss: $f(x) = \lambda \max(0, 1 - x)$.

Using these answers, write the iterates update of PGD algorithm for a composite function of the form:

$$f = g + h,$$

where:

1. Lasso regression: $g(x) = \|\mathbf{A}x - b\|_2^2, h(x) = \|x\|_1$.
2. Group Lasso regression: $g(x) = \|\mathbf{A}x - b\|_2^2, h(x) = \sum_{i=1}^K \|x_{\mathcal{I}_i}\|_2$.

Exercise 4 (Conjugate function). Consider a function $f : \mathbb{R}^d \rightarrow \mathbb{R} \cup \{+\infty\}$, we define its conjugate as:

$$f^*(y) = \sup_{x \in \mathbb{R}^d} y^\top x - f(x)$$

Prove the following claims:

1. Fenchel's inequality: $f(x) + f^*(y) \geq x^\top y, \forall x, y \in \mathbb{R}^d$.
2. f^* is lower semicontinuous and convex for any function f .
3. The second conjugate: let f^{**} be the conjugate function of f^* (which is itself a conjugate function of f). If f is lower semicontinuous and convex, then $f^{**} = f$ (Hint: proof by contradiction, and use the separating plane theorem).
4. If f is lower semicontinuous and convex, then:

$$y \in \partial f(x) \iff x \in \partial f^*(y) \iff x^\top y = f(x) + f^*(y).$$

Exercise 5 (Firm non-expansivity of proximal operators). Let $f : \mathbb{R}^d \rightarrow \mathbb{R}$ be a lower semicontinuous, proper and convex function. Prove that for any $x, y \in \mathbb{R}^d$, we have:

$$\|\text{prox}_f(x) - \text{prox}_f(y)\|^2 \leq \langle x - y, \text{prox}_f(x) - \text{prox}_f(y) \rangle.$$

Deduct that proximal operators are non-expansive, i.e.,

$$\|\text{prox}_f(x) - \text{prox}_f(y)\| \leq \|x - y\|.$$

Exercise 6 (Moreau decomposition theorem). Let f be a lower semicontinuous, proper and convex function. Prove that:

$$x = \text{prox}_f(x) + \text{prox}_{f^*}(x).$$

As a consequence of Moreau decomposition theorem, prove that for $f(x) = \|x\|_\infty$, we have:

$$\text{prox}_f(x) = x - \text{proj}_{\|\cdot\|_1 \leq 1}(x),$$

where $\text{proj}_{\|\cdot\|_1 \leq 1}$ is the projection onto the unit ℓ_1 ball.

Exercise 7 (Convergence in nonconvex settings). We already see the theoretical properties in the convex case. We investigate the non-convex case in this setting. Consider a composite function:

$$f = g + h,$$

where:

1. g is L -smooth function.
2. h is proper and lower semicontinuous function. We assume in addition that $\text{prox}_{\alpha h}$ is well-defined for all $\alpha > 0$ (e.g., if h is additionally convex).
3. f is lower bounded below.

Prove that for a constant step-size $0 < \alpha \leq 1/L$, we have:

$$\lim_{k \rightarrow \infty} \left\| \frac{1}{\alpha} (x_k - \text{prox}_{\alpha h}(x_k - \alpha \nabla g(x_k))) \right\|^2 = O\left(\frac{1}{k}\right).$$