Tutorial 2 : Convex sets and functions

Exercise 1 (Unproven proposition)**.** If A is a convex set, then any convex combination of finite points of A is an element of A, i.e., for $x_1, \ldots, x_k \in A$, $\alpha_1, \ldots, \alpha_k \geq 0, \sum_i \alpha_i = 1$, we have:

$$\alpha_1 x_1 + \ldots + \alpha_k x_k \in A.$$

Exercise 2 (Convex sets)**.** Decide whether the following sets are convex and justify your choice:

1. A slab, i.e., a set of the form: $\{x \in \mathbb{R}^n \mid \alpha \leq x^\top a \leq \beta\}$ (α, β, a are fixed scalars and vector).
2. A rectangle, i.e., a set of the form: $\{x \in \mathbb{R}^d \mid \alpha_i \leq x_i \leq \beta_i\}$.
3. A wedge, i.e., $\{x \in \mathbb{R}^d \mid a_1^\top x \leq b_1, a_2^\top x \leq b_2\}$.
4. The set of points closer to a given point than a given set, i.e.,
$$\{x \in \mathbb{R}^d \mid \|x - x_0\|_2 \leq \|x - y\|_2, \forall y \in S\}$$
5. The set of points closer to one set than another, i.e.,
$$\{x \in \mathbb{R}^d \mid \text{dist}(x, S) \leq \text{dist}(x, T)\}$$
where $\text{dist}(x, S) = \inf_{y \in S} \|x - y\|_2$.
6. The set $\{x \in \mathbb{R}^d \mid x + S_2 \subseteq S_1\}$ where $S_1, S_2 \subseteq \mathbb{R}^d$ and S_1 is convex.
7. The set of points whose distance to a does not exceed a fixed fraction θ of the distance to b, i.e., $\{x \in \mathbb{R}^d \mid \|x - a\|_2 \leq \theta \|x - b\|_2\}$. You can assume that $a \neq b$ and $\theta \in [0, 1]$.

Exercise 3 (Operations preserving convexity)**.** Prove that the following operations preserve the convexity:

1. If f is convex, then αf is also convex ($\alpha > 0$).
2. If f and g are convex, then $f + g$ is also convex.
3. If $f_i, i = 1, \ldots, n$ are convex and $\alpha_i \geq 0, \forall i = 1, \ldots, n$, then $\sum_i \alpha_i f_i$ is also convex.
4. If f, g are convex, then $\max(f, g)$ is convex.
5. If f is convex, then $g(x) = f(Ax + b)$ is also convex.

6. if $f : \mathbb{R}^d \to \mathbb{R}$ is convex and $g : \mathbb{R} \to \mathbb{R}$ is non-decreasing and convex, then $g \circ f$ is convex.

Proof. To be done. □

Exercise 4 (Sublevel sets of convex functions)**.** Given a function $f : \mathbb{R}^d \to \mathbb{R}$, the α, the sublevel sets of f is given by:

$$\{x \in \mathbb{R}^d \mid f(x) \leq c\}.$$

1. Prove that the sublevel set of a convex function is convex.
2. Is the converse correct, i.e., if all the sublevels of f are convex, is f necessarily a convex function?
3. Applying this result to show that the ellipsoid, i.e., $\{x \in \mathbb{R}^d \mid (x-y)^\top \mathbf{A}(x-y) \leq c\}$ for some $\mathbf{A} \in \mathbb{R}^{d\times d}, \mathbf{A} \succeq 0, y \in \mathbb{R}^d, c \in \mathbb{R}$.

Exercise 5 (Optimal condition for constrained convex optimization)**.** Consider a constrained optimization problem given by:

$$\underset{x \in \mathcal{F}}{\text{Minimize}} \quad f(x)$$

where f is C^1 and convex, $\mathcal{F}$ is a convex set. A point x is a global solution of $f(x)$ if and only if $\nabla f(x)^\top (y-x) \geq 0, \forall y \in \mathcal{F}$.

Exercise 6 (Projection onto a convex set)**.** In this exercise, we will prove for any non-empty closed convex set $S \subseteq \mathbb{R}^d, x \in \mathbb{R}^d$, the set of projector of x onto S is a singleton, i.e., $\{y \in S \mid \|x-y\|_2 = \text{dist}(x,S)\}$. To do so, answer the following questions:

1. Prove that the set of projectors of x onto S is non-empty.
2. Let y be a projector of x onto S, prove that:

   $$(y-x)^\top (z-y) \geq 0, \forall z \in S.$$

   Hint: Use the previous exercise.
3. Prove that the set of projectors has at most one element by contradiction.

Proof. Proof is left as exercise. □

Exercise 7 (Seperating plane of disjoint convex sets)**.** In this exercise, we will prove the following results: For any two non-empty, convex sets $C, D \subseteq \mathbb{R}^d$ such that $C \cap D = \emptyset$, there exists a separating hyperplane, i.e., a vector $a \in \mathbb{R}^d, b \in \mathbb{R}$ satisfying:

$$x \in C \implies x^\top a \leq b,$$
$$x \in D \implies x^\top a \geq b.$$

Prove a simpler version of this result where we assume that there exists $c \in C, d \in D$ such that $\|x-y\|_2 = \text{dist}(C,D)$ by following these steps:

1. Define $a = d - c$, $b = \frac{\|d\|^2 - \|c\|^2}{2}$. Prove that:

   $$f(x) = x^\top a - b = (d-c)^\top \left(x - \frac{1}{2}(d+c)\right) = \frac{1}{2}\|d-c\|^2 + (d-c)^\top (x-d).$$

2. Prove that for $x \in D$, we have: $(d-c)^\top (x-d) \geq 0$.
3. Conclude that for $x \in D$, $x^\top a \geq b$. Conclude the proof by making the same argument can be used to prove that $x \in C \implies x^\top a \leq b$.