Tutorial 11 : Linear Programming and Integer Linear Program

Exercise 1 (Existence of optimal solutions)**.** Prove that if an instance of linear programming is feasible and its optimal value is finite, then it admits an optimal solution.

Exercise 2 (Dual problems calculation)**.** Compute the dual problems of the following integer programming instances:

1.

$$\begin{array}{lllll} \text{Minimize} & 4x_1 & +3x_2 & +2x_3 & \\ \text{subject to} & 2x_1 & +x_2 & +x_3 & \geq 10 \\ & x_1 & -3x_2 & +2x_3 & \geq 15 \\ & 2x_1 & +2x_2 & -3x_3 & \geq 18 \\ & & & x & \geq 0 \end{array}$$

2.

$$\begin{array}{lllll} \text{Minimize} & x_1 & +3x_2 & +2x_3 & \\ \text{subject to} & 2x_1 & +x_2 & +x_3 & \geq 5 \\ & x_1 & -3x_2 & +2x_3 & \leq 7 \\ & 2x_1 & +2x_2 & -3x_3 & = 3 \\ & & & x_3 & \geq 0 \end{array}$$

3.

$$\begin{array}{lllll} \text{Maximize} & 6x_1 & -2x_2 & +5x_3 & \\ \text{subject to} & x_1 & -4x_2 & +x_3 & \geq 5 \\ & 2x_1 & -2x_2 & -5x_3 & = 7 \\ & 4x_1 & +3x_2 & -7x_3 & = 3 \\ & & & x_1, x_3 & \leq 0 \\ & & & x_2 & \geq 0 \end{array}$$

Exercise 3 (Deduce the optimal solutions from primal/dual results)**.** Consider a pair of primal and dual problems:

$$\begin{array}{lllll} \text{Minimize} & 5x_1 & +11x_2 & +8x_3 & \\ \text{subject to} & 2x_1 & +4x_2 & +3x_3 & \geq 5 \\ & 3x_1 & -x_2 & +4x_3 & \geq 4 \\ & x_1 & +2x_2 & -2x_3 & \geq 3 \\ & & & x & \geq 0 \end{array} \tag{P}$$

$$\begin{array}{llllll} \text{Maximize} & 5y_1 & +4y_2 & +3y_3 & & \\ \text{subject to} & 2y_1 & +3y_2 & +y_3 & \geq 5 & \\ & 4y_1 & -y_2 & +2y_3 & \leq 11 & \\ & 3y_1 & +4y_2 & -2y_3 & = 8 & \\ & & & y & \geq 0 & \end{array} \tag{D}$$

Knowing that $(2, 0, 1)$ is an optimal solution for (P), find the optimal solution of (D).

Exercise 4 (Farkas lemma resivited)**.** Remind the Farkas lemma: given $\mathbf{B} \in \mathbb{R}^{n\times m}$ and $\mathbf{C} \in \mathbb{R}^{n\times p}$. Define the cone:

$$K := \{\mathbf{B}y + \mathbf{C}w \mid y \in \mathbb{R}^m_+, w \in \mathbb{R}^p\}.$$

Prove that only one of the two alternatives is true:

1. $g \in K$.
2. There exists $d \in \mathbb{R}^n$ such that $g^\top d < 0$, $\mathbf{B}^\top d \geq 0, \mathbf{C}^\top d = 0$.

Provide a proof for the Farkas lemma using linear programming.

Exercise 5 (ILP formulation)**.** Write the ILP formulations for the following problems:

1. **Independent Set problem**: Given a graph $G = (V, E)$, an independent set is the set of vertices pairwise not incident. Formulate the Maximum Independent Set.
2. **Dominating Set Problem**: Given a graph $G = (V, E)$, a dominating set is the set of vertices X such that any vertex of the graph is either in X or incident to a vertex of X. Formulate the Minimum Donimating Set problem.
3. **N-queens problem** Suppose that you are given an $N \times N$-chessboard. Formulate the problem of placing N queens on the board such that no two queens share any row, column and diagonal.
4. **Traveling Saleman Problem (TSP)** Given a weighted complete graph G. The goal consists in finding a cycle passing exactly once through all the vertices of the graph such that the sum of its edge weights is minimize. Formulate this problem.

Exercise 6 (Minimum Weighted Set Cover)**.** Given $N = \{1, \ldots, n\}, T_i \subseteq N, i = 1, \ldots, r$ and $w_i \geq 0, i = 1, \ldots, r$ a weight/cost associated to T_i, a set cover S is subset of $\{T_i \mid i = 1, \ldots, r\}$ satisfying $\cup_{T\in S} T = N$. The Minimum Weighted Set Cover problem asks to find a set cover that minimize their total cost.

1. Write an ILP to solve the problem.
2. Let $x^\star$ be an optimal solution of the relaxation. Let us set $x_i = 1$ with probability $x_i^\star$ and 0 otherwise. Prove that the expected value of the total cost is the optimal value of the relaxed LP problem.
3. Prove that the probability of a fixed index j uncovered by x (sampled by the strategy as the previous question) is at most $1/e$.
4. Deduce an $O(\log n)$ approximation algorithm (with constant failure probability), i.e., there exists $0 \leq c < 1$ such that the obtained solutions yielding a value at most $O(\log n)$ times the true optimal one (hint: use Markov inequality).