Tutorial 10 : Theory of constrained optimization

Exercise 1 (Cones defined by linear transformation)**.** Given $\mathbf{B} \in \mathbb{R}^{n\times m}$ and $\mathbf{C} \in \mathbb{R}^{n\times p}$. Define the cone:

$$K := \{\mathbf{B}y + \mathbf{C}w \mid y \in \mathbb{R}^m_+, w \in \mathbb{R}^p\}.$$

Prove that K is a closed, convex cone (remind that a set is a cone if for $x \in K$, then $\alpha x \in K, \forall \alpha \geq 0$).

Note: You can assume that linear transformation of an intersection of finite half-spaces remains an intersection of finite half-spaces.

Exercise 2 (Farkas lemma)**.** Consider the set K in the previous exercise. Prove that only one of the two alternatives is true:

1. $g \in K$.
2. There exists $d \in \mathbb{R}^n$ such that $g^\top d < 0$, $\mathbf{B}^\top d \geq 0, \mathbf{C}^\top d = 0$.

Hint: use the previous exercise.

Exercise 3 (Finding optimal solutions with KKT condition)**.** Solving the following constrained optimization problem with KKT conditions:

1. Minimize $f(x_1, x_2) = 5x_1 - 3x_2$ subject to $x_1^2 + x_2^2 = 1$.
2. Minimize $f(x_1, x_2) = x_1 x_2$ subject to $ax_1^2 + x_2^2 = 1$ (a is a fixed constant, $a \geq 0$).
3. Maximize $f(x_1, x_2) = x_1 x_2$ subject to $ax_1^2 + x_2^2 \leq 1$ (a is a fixed constant, $a \geq 0$).

Exercise 4 (Dual functions computation)**.** Compute the dual problems of the following constrained optimization problems:

1. Inequality form of linear programming:

$$\begin{aligned} \underset{x}{\text{Minimize}} \quad & c^\top x \\ \text{subject to:} \quad & \mathbf{A}x \leq b, \end{aligned}$$

2. Equality constrained norm minimzation:

$$\begin{aligned} \underset{x}{\text{Minimize}} \quad & \|x\|_2 \\ \text{subject to:} \quad & \mathbf{A}x = b, \end{aligned}$$

3. Entropy maximization:

$$\begin{aligned} \underset{x}{\text{Minimize}} \quad & f(x) = \sum_{i=1}^{n} x_i \log x_i \\ \text{subject to:} \quad & \mathbf{A}x \leq b, \\ & \sum_{i=1}^{n} x_i = 1. \end{aligned}$$

4. Unconstrained geometric program:

$$\underset{x}{\text{Minimize}} \quad \log(\sum_{i=1}^{m} \exp(a_i^\top x + b_i))$$

Exercise 5 (LICQ constraint qualifications)**.** Prove that when the KKT conditions and the LICQ are satisfied at a point $x^\star$, then the Lagrangre multiplier $\lambda^\star$ is unique.

Exercise 6 (KKT are sufficient in the convex and affine equality case)**.** Let $c_i, i \in \mathcal{E}$ be linear and $f, c_i, i \in \mathcal{I}$ be differentiable and convex functions (in this exercise, $c_i(x) \leq 0, \forall i \in \mathcal{I}$). Prove that if there exists $x^\star, \lambda^\star, \nu^\star$ satisfying the KKT conditions, then $x^\star$ is an optimal solution of the constrained problem.

Exercise 7 (LICQ implies $T_{\mathcal{F}}(x) = \mathcal{F}(x)$)**.** Remind that $\mathcal{F}(x)$ and $T_{\mathcal{F}}(x)$ are the linearized tangent cone and the tangent cone respectively. Prove that if LICQ holds at x, then $T_{\mathcal{F}}(x) = \mathcal{F}(x)$.